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Scaling Analysis for the Adsorption Transition in a Watermelon Network of *n* Directed Non-Intersecting Walks[1]

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The partition function for the problem of n directed non-intersecting walks interacting via contact potentials with a wall parallel to the direction of the walks has previously been calculated as an $n \times n$ determinant. Here, we describe how to analyse the scaling behaviour of this problem using alternative representations of the solution. In doing so we derive the asymptotics of the partition function of a watermelon network of n such walks for all temperatures, and so calculate the associated network exponents in the three regimes: desorbed, adsorbed, and at the adsorption transition. Furthermore, we derive the full scaling function around the adsorption transition for all n. At the adsorption transition we also derive a simple "product form" for the partition function. These results have, in part, been derived using recurrence relations satisfied by the original determinantal solution.

KEY WORDS: Vicious walkers; directed walks; lattice paths; interacting self-avoiding walks; adsorption transition; watermelon network.

1. INTRODUCTION

The problem of n (fully) directed walks interacting with a wall via contact interactions on the square lattice is a non-trivial example of a statistical mechanical system showing a phase transition that is exactly solvable for finite size as well as in the thermodynamic limit. In addition these lattice

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path problems (as opposed to their continuous analogues) are of considerable significance in enumerative and constructive combinatorics.[1,2] As such the set of problems for all n represent an infinite hierarchy of models, each individually solvable in the full two parameter space of temperature and system size. In fact there are different variants of these problems depending on the restrictions placed on the end-points of the walks. Much is already known about these problems for small n and some general solutions are known for all n. However the analysis of the functions involved in the exact solution has not been carried out. In this paper we provide a full analysis of the scaling behaviour of this set of problems in the case of one particular type of end-point restrictions.

With $n=1$ and $n=2$ exact solutions for the partition functions of fixed length, with various standard end-point conformations, have been calculated.[3] In those cases, if one weights each of the contacts with the wall by a Boltzmann factor, κ, then there is an adsorption phase transition at a value $\kappa=\kappa_c=2$. On the other hand with $\kappa=1$, that is, the non-interacting case,[4] much is known about the solutions for all n. This is also true when there is no wall present[5] at all, known as the bulk case. The exact solution has been calculated for both the non-interacting and bulk cases: it can be found from the Gessel–Viennot theorem,[6] when the endpoints of the walks are fixed, as a determinant.[7,6,8,9] In fact, the determinants, or sums over determinants if the endpoints are summed over, can be evaluated as products of (ratios of) factorials. From such expressions it is a straightforward matter to calculate the asymptotics of the partition function, and so evaluate both the connective constant, which is related to the free energy (or rather the entropy) and the network exponents, γ_n, for the topology concerned. "Product forms",[10,11,8,9] which have deep combinatorial significance, and network exponents[4,11] can be calculated for both the standard watermelon and star conformations when a wall is present, without additional interactions, and also in the bulk. The product forms can be found by direct determinantal manipulation[11] or by using mappings to Young Tableaux and plane partitions.[8]

An important recent development[12] has been the calculation of the partition function of the interacting problem for all κ and for arbitrary n, with fixed endpoints, in terms of an $n\times n$ determinant of $n=1$ partition functions. In this way the solution becomes a sum over $n!$ terms each of which are products of n weighted sums of binomial coefficients. The scaling analysis for fixed κ is therefore problematic due to the complexity of this expression. Some hope of progress is called for since in the case of $n=2$ the determinantal expression has be simplified[3] to what is, essentially, a single sum. This single sum allows meaningful analysis. The solution in the case $n=2$ can also be seen to obey a single difference equation (or recurrence relation). Importantly, this difference equation can be analysed for the asymptotic behaviour of the partition function without appealing to the solution itself.[3]

For arbitrary n various combinatorial properties of coefficients in expansions of the partition function have now been found.[13] In particular, some of these considerations lead to two different recurrence relations for the partition functions. These recurrences are two variable recurrences in n and t, where t is the length of the walks. Here we utilise these combinatorial properties and the recurrences to derive a single variable recurrence in the length t for each n; this being the generalisation of the recurrences mentioned above for $n=1$ and $n=2$. We are then able to provide an analysis of the dominant asymptotic behaviour of the partition function scaling for all fixed κ and for any n. We also are able to derive a "product form" for the partition function at $\kappa=2$: we argue additionally that such a form is unlikely to exist for all values of κ, although we point out that the coefficients of expansions of the partition function can be written as products for all κ.[13]

Finally, we use one such expansion to calculate an infinite hierarchy of scaling functions which describe the dominant two variable asymptotics around the phase transition point. We point out that an exact scaling function for one, let alone an infinite set of problems, is rare in the literature.

2. DIRECTED-WALK SURFACE-WATERMELONS

We begin by considering the semi-infinite, or half-plane, square lattice which has been rotated 45° (see Fig. 1). A directed lattice path, or walk, is a set of occupied sites connected by bonds of the lattice. Moreover, the walks have steps (the connecting lattice bonds, which are also considered occupied) in only the north-east or south-east directions. We shall label the site (s, y) in column s and row y as in Fig. 1. A set of walks is *non-intersecting* if they have no sites in common. We are concerned with configurations of n non-intersecting walks, starting and ending at given positions in common columns of the lattice, each walk being of length t. A walk of "length" t contains $t+1$ occupied sites. Let $y_j(s)$, $j=1,..., n$ be the positions of the walks in column s. Non-intersection implies that $y_j(s) < y_{j+1}(s)$, $j=1,...,(n-1)$ for all $0 \leqslant s \leqslant t$. The half-plane restriction implies that $y_j(s) \geqslant 0$ for all s and a "wall" is considered to exist below the height (or row) $y=0$.

In this paper we shall restrict ourselves, for the sake of clarity, to so-called watermelon configurations that start and end on the surface. We shall call these "surface-watermelons." To be precise if $y_j(0) = y_j^i$, $j=1,..., N$ are the positions of the walks in column $s=0$ and $y_j(t) = y_j^f$, $j=1,..., N$ are the positions of the walks in column $s=t$ then our watermelons are those

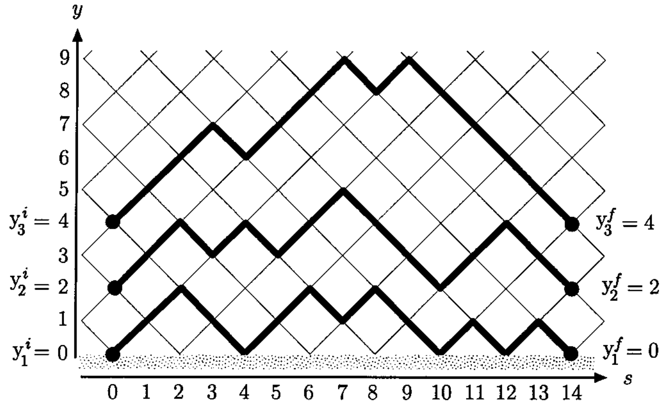

Fig. 1. Three non-intersecting directed walks of length $t = 14$ in a watermelon configuration tied to the surface, where $y_j^i = y_j^f = 2(j-1)$. The coordinates for the rows y and columns s are shown. Also illustrated are the positions of the walks in the columns $s = 0$ and $s = t$ which are denoted by y^i and y^f respectively. The walk closest to the lower wall has weight κ^5.

where $y_j^i = y_j^f = 2(j-1)$. Note that due to the lattice such configurations are only possible if the numbers of columns spanned by the walks, t, is even. Odd lengths can be considered by a minor modifications of the results given below. Hence, we shall only consider walks of length $t = 2r$.

The partition function of n non-intersecting walks interacting with a wall being in a surface-watermelon configuration, is defined as

$$\hat{Z}_{2r}^{(n)}(\kappa) = \sum_m c_{2r}^{(n)}(m)\, \kappa^m \tag{2.1}$$

where $c_{2r}^{(n)}(m)$ is the number of configurations of n-walk surface-watermelons of length $2r$ with m sites occupied in row $y = 0$ of the lattice. It has been shown that the partition function, $\hat{Z}_{2r}^{(n)}(\kappa)$, of n non-intersecting walks in a surface-watermelon configuration is given by the following determinant:

$$\hat{Z}_{2r}^{(n)}(\kappa) = \begin{vmatrix} Z_{2r}^S(0 \to 0; \kappa) & Z_{2r}^S(0 \to 2; \kappa) & \cdots & Z_{2r}^S(0 \to 2n-2; \kappa) \\ Z_{2r}^S(2 \to 0; \kappa) & Z_{2r}^S(2 \to 2; \kappa) & \cdots & Z_{2r}^S(2 \to 2n-2; \kappa) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ Z_{2r}^S(2n-2 \to 0; \kappa) & Z_{2r}^S(2n-2 \to 2; \kappa) & \cdots & Z_{2r}^S(2n-2 \to 2n-2; \kappa) \end{vmatrix} \tag{2.2}$$

where $Z^S_{2r}(2x \to 2y; \kappa)$ is the partition function for configurations of a single walk starting at height $2x$ in column 0 and ending at height $2y$ in column $2r$ in the presence of a wall. The single walk partition function, $Z^S_{2r}(2x \to 2y; \kappa)$, has been calculated[3] as

$$Z^S_{2r}(2x \to 2y; \kappa) = \binom{2r}{r+x-y} - \binom{2r}{r-x-y} + \kappa \sum_{\ell \geqslant 0} (\kappa-1)^\ell \left\{ \binom{2r}{r-\ell-x-y} - \binom{2r}{r-\ell-x-y-1} \right\} \tag{2.3}$$

Note that $Z^S_{2r}(0 \to 0; \kappa) \equiv \hat{Z}^{(1)}_{2r}(\kappa)$.

3. A REVIEW OF A SINGLE DIRECTED WALK ATTACHED TO A WALL AT BOTH ENDS

Let us first consider the known results about $n = 1$. The partition function can be written as an expansion in three different variables. Firstly, using (2.3) we can write[3]

$$\hat{Z}^{(1)}_{2r}(\kappa) = \kappa \sum_{\ell=0}^{r} (\kappa-1)^\ell \left\{ \binom{2r}{r-\ell} - \binom{2r}{r-\ell-1} \right\} \tag{3.4}$$

which is a polynomial in $(\kappa - 1)$. However, there are other expressions for this quantity: secondly, it can be written as an expansion in the variable κ

$$\hat{Z}^{(1)}_{2r}(\kappa) = \sum_{m=0}^{r+1} \kappa^m \left\{ \binom{2r-m}{r-1} - \binom{2r-m}{r} \right\} \tag{3.5}$$

Related to this expansion is the generating function of the partition functions (or rather generalised partition function)

$$\hat{G}^{(1)}(\kappa; z) = \sum_{r=0}^{\infty} \hat{Z}^{(1)}_{2r}(\kappa)\, z^{2r} \tag{3.6}$$

which can be evaluated as

$$\hat{G}^{(1)}(\kappa; z) = \frac{\kappa}{1 - (\kappa/2)(1 - \sqrt{1-4z^2})} \tag{3.7}$$

Thirdly, there is also an expansion in the variable

$$\omega(\kappa)=\frac{\kappa-1}{\kappa^2} \tag{3.8}$$

which is

$$\hat{Z}_{2r}^{(1)}(\kappa(\omega))=\frac{\kappa(\kappa-2)}{\kappa-1}\,\omega^{-r}\theta(\kappa-2)+\sum_{s=r}^{\infty} C_s\omega^{s-r} \tag{3.9}$$

where $\theta(x)$ is a unit step function and C_s is the sth Catalan number,

$$C_s=\frac{1}{s+1}\binom{2s}{s} \tag{3.10}$$

While somewhat mysterious from the point of view of the walk problem the variable ω is the "natural" variable when modelling directed compact percolation.[14] It is this form that is most useful in the calculation of the full scaling form around the transition point.

There are two special non-zero and finite values of κ where the partition function can be rewritten as a product. When $\kappa=1$

$$\hat{Z}_{2r}^{(1)}(1)=4^r\prod_{j=1}^{r}\frac{2j-1}{2j+2} \tag{3.11}$$

and when $\kappa=2$

$$\hat{Z}_{2r}^{(1)}(2)=2\,4^r\prod_{j=1}^{r}\frac{2j-1}{2j} \tag{3.12}$$

It is unlikely that the partition function can be written as such a finite product of "simple" real linear factors for all values of κ since the zeros of the partition function of length $2r-2$ and $2r$ would need to coincide for most of the zeros: this is not the case even for $r=4$ for example. This rough argument does not, of course, rule out other "special" values of κ for which their may be product forms.

Importantly the partition function satisfies an inhomogeneous first-order recurrence (see Eq. (3.29) of ref. 3) which can be converted to the following homogeneous second-order recurrence

$$\omega r\hat{Z}_{2r}^{(1)}-[r+(4r-6)\,\omega]\,\hat{Z}_{2r-2}^{(1)}+(4r-6)\,\hat{Z}_{2r-4}^{(1)}=0 \tag{3.13}$$

Let us now briefly discuss how each of the above expressions can be used to ascertain the salient features of the asymptotics of the partition function in the large r limit. In fact, the simplest method to analyse this problem is through analysis of the singularities of the generating function and then use of Darboux's Theorem to imply the asymptotic behaviour of the partition function. For $0<\kappa<2$ the generating function has an algebraic square root singularity at the value $z_c=1/2$, independent of κ, on the positive real axis

$$\hat{G}^{(1)}(\kappa; z) \sim A \sqrt{1-\frac{z}{z_c}} \qquad \text{as} \quad z \to z_c \tag{3.14}$$

For $\kappa>2$ there is a simple pole in the generating function at

$$z_c(\kappa)=\sqrt{\omega(\kappa)} \tag{3.15}$$

At $\kappa=2$ the generating function has a divergent square root singularity at $z_c=1/2$. Hence we can deduce that the partition function has the asymptotic form

$$\hat{Z}_{2r}^{(1)}(\kappa) \sim B_1 \mu_1^{2r} r^{g_{11}^{(1)}} \qquad \text{as} \quad r \to \infty \tag{3.16}$$

with

$$\mu_1(\kappa)=\begin{cases} 2 & \text{if} \quad 0<\kappa\leqslant 2 \\ \dfrac{1}{\sqrt{\omega(\kappa)}} & \text{if} \quad \kappa>2 \end{cases} \tag{3.17}$$

noting that $\sqrt{\omega(\kappa)} \to 1/2$ as $\kappa \to 2$ so that $\mu_1(\kappa)$ is continuous, and

$$g_{11}^{(1)}=\begin{cases} -3/2 & \text{if} \quad 0<\kappa<2 \\ -1/2 & \text{if} \quad \kappa=2 \\ 0 & \text{if} \quad \kappa>2 \end{cases} \tag{3.18}$$

The factor B_1 is a function of κ. The thermodynamic-limit (reduced) free energy defined as

$$f^{(1)}(\kappa)=\lim_{r\to\infty} \frac{1}{2r} \log(\hat{Z}_{2r}^{(1)}(\kappa)) \tag{3.19}$$

therefore exists and

$$f^{(1)}(\kappa)=\log \mu_1(\kappa) \tag{3.20}$$

This implies via (3.17) that the thermodynamic exponent $\alpha = 0$ and that the associated specific heat has a jump discontinuity.

If one were to attempt to analyse the partition function expressions directly, the product forms (3.11) and (3.12), only available at $\kappa = 1$ and $\kappa = 2$, are the simplest to tackle, requiring essentially only Stirling's approximation. Of the three expansions in $\kappa, \kappa - 1$ and ω it is, curiously enough, the expansion in ω that is the most useful to analyse. It gives the scaling form around the phase transition point $\kappa = 2$ (see Section 3.6.1 of ref. 3): this has been found as

$$\hat{Z}_{2r}^{(1)}(\kappa) \approx \frac{4^r}{r^{1/2}} \hat{\varphi}^{(1)}\left(\frac{(\kappa - 2)}{2} r^{1/2}\right) \qquad \text{as} \quad \kappa \to 2 \quad \text{and} \quad r \to \infty \tag{3.21}$$

with

$$\hat{\varphi}^{(1)}(v) = \frac{2}{\sqrt{\pi}} + 2ve^{v^2} \operatorname{erfc}(-v) \tag{3.22}$$

The notation $\approx$ refers to the well-defined set[15] of assumptions associated with the use of crossover scaling asymptotic forms. This is in contrast to the standard, but ill-defined, use of $\sim$ in this context (as mentioned by ref. 16). At the risk of oversimplification, $\approx$ adds to $\sim$ the necessary asymptotic uniformity and matching conditions for what is essentially a two variable asymptotic statement. The scaling variable

$$v = \frac{(\kappa - 2)}{2} r^{1/2} \tag{3.23}$$

gives us immediately that the crossover exponent, ϕ, of the transition is 1/2.

Lastly, but importantly, the quickest way to find the results (3.17) and (3.18) without using the generating function is to analyse the recurrence (3.13) using a dominant balance method[3] (see Section 3.2 and Appendix 1 of that paper).

4. SCALING ANALYSIS OF THE n-WALK PARTITION FUNCTION

For n directed walks we expect that

$$\hat{Z}_{2r}^{(n)}(\kappa) \sim B_n \mu_n^{2r} r^{g_{11}^{(n)}} \qquad \text{as} \quad r \to \infty \tag{4.24}$$

where B_n will be a function of κ. For $n=1$ the value of the connective constant μ_1 and $g_{11}^{(1)}$ exponent are given above. Previous work by Forrester[4] has found at $\kappa=1$ that $\mu_n = 2^n$ and

$$g_{11}^{(n)} = -\frac{n(2n+1)}{2} \tag{4.25}$$

The exponents $g_{11}^{(n)}$ are the so-called network exponents for this problem. In this paper we shall calculate network exponents for all values of κ.

It may be possible to extend the use of the generating function approach to arbitrary numbers of walks, though we point out that going from one to two walks changes the generating function from a simple algebraic function to a sum of generalised hypergeometric ones. We rather have taken the more direct route of considering the partition function, or equations it satisfies, to find its asymptotic behavior.

4.1. Bounds on the Thermodynamic-Limit Free Energy

Standard rigorous arguments[17] can be used to show that the free energy

$$f^{(n)}(\kappa) = \lim_{r \to \infty} \frac{1}{2r} \log(\hat{Z}_{2r}^{(n)}(\kappa)) \tag{4.26}$$

exists for all $\kappa > 0$ and that

$$f^{(n)}(\kappa) \geqslant f^{(n)}(1) \tag{4.27}$$

In fact one can show

$$f^{(n)}(\kappa) = f^{(n)}(1) = n \log 2 \qquad \text{for} \quad 0 < \kappa \leqslant 1 \tag{4.28}$$

The reduced free energy is a continuous and non-decreasing function of κ. Other bounds can be found from standard arguments as

$$f^{(n)}(1) + \log(\kappa^{1/2}) \geqslant f^{(n)}(\kappa) \geqslant f^{(n-1)}(1) + \log(\kappa^{1/2}) \tag{4.29}$$

for $\kappa \geqslant 1$. Putting all this information together implies that the free energy has at least one singularity as a function of κ, and that the first singularity must occur for some $1 < \kappa \leqslant 4$.

4.2. Network Recurrence Relation

In recent work[13] on the combinatorics of this problem the following two recurrence relations for the partition function were derived:

$$\omega \hat{Z}_{2r-2}^{(n-1)}(1)\, \hat{Z}_{2r}^{(n)}(\kappa) = \hat{Z}_{2r}^{(n-1)}(1)\, \hat{Z}_{2r-2}^{(n)}(\kappa) - \kappa^{-2} \hat{Z}_{2r-2}^{(n)}(1)\, \hat{Z}_{2r}^{(n-1)}(\kappa) \tag{4.30}$$

and

$$\hat{Z}_{2r}^{(n)}(\kappa) = \kappa^{-2} [\hat{Z}_{2r}^{(n-1)}(1)\, \hat{Z}_{2r+4}^{(n-1)}(\kappa) - \hat{Z}_{2r+2}^{(n-1)}(1)\, \hat{Z}_{2r+2}^{(n-1)}(\kappa)] / \hat{Z}_{2r+4}^{(n-2)}(1) \tag{4.31}$$

In this derivation all walks were first extended back to the surface so the determinant corresponding to (2.2) had elements of the type $\hat{Z}_{2r}^{(1)}(\kappa)$. The most important step was to use a first order recurrence for $\hat{Z}_{2r}^{(1)}(\kappa)$ to remove the κ dependence from all but the last column of the determinant. The second of the above relations was then obtained by application of Dodgson's condensation formula[18] to the resulting determinant. The first was found empirically and proved using a product formula for the coefficients in an ω expansion of $\hat{Z}_{2r}^{(n)}(\kappa)$ found by expanding the last column of the modified determinant in powers of ω.

Each of these recurrences contain shifts in both the length $2r$ and numbers of walks involved n. However they can be combined to give

$$\begin{aligned} &\omega \hat{Z}_{2r+4}^{(n)}(\kappa) - (\omega \rho_r^{(n)} + \rho_{r+1}^{(n-1)})\, \hat{Z}_{2r+2}^{(n)}(\kappa) \\ &\quad + \rho_{r-1}^{(n)} \rho_{r+1}^{(n-1)} \left(1 + \frac{\xi_{r-1}^{(n)}}{\xi_r^{(n-1)}}\right) \hat{Z}_{2r}^{(n)}(\kappa) = 0 \end{aligned} \tag{4.32}$$

where

$$\rho_r^{(n)} = \frac{\hat{Z}_{2r+2}^{(n)}(1)}{\hat{Z}_{2r}^{(n)}(1)} \tag{4.33}$$

and

$$\xi_r^{(n)} = \frac{\hat{Z}_{2r}^{(n+1)}(1)}{\hat{Z}_{2r+2}^{(n)}(1)} \tag{4.34}$$

As mentioned previously the case $\kappa = 1$ has been analysed extensively and a simple product form[11] for the partition function, derived by elementary manipulations of the determinantal form, is known

$$\hat{Z}_{2r}^{(n)}(1) = \prod_{i=1}^{n} \frac{(2r+2i-2)!\,(2i-1)!}{(r+i-1)!\,(r+i+n-1)!} \tag{4.35}$$

Hence

$$\rho_r^{(n)} = \frac{(2r+1)_{2n}}{(r+1)_{2n}} \tag{4.36}$$

and

$$\xi_r^{(n)} = \frac{(r+1)_r}{(2n+2)_r} \tag{4.37}$$

where $(a)_n = \prod_{j=1}^{n}(a+j-1) = \Gamma(a+n)/\Gamma(a)$ with $(a)_0 \equiv 1$ is the Pochhammer symbol. This leads here to the explicit form of the recurrence relation above as

$$\begin{aligned} &\omega(r+2n-2)(r+n-1)_{n-1}^2\, \hat{Z}_{2r}^{(n)}(\kappa) \\ &\quad -4^{n-1}(r-\tfrac{1}{2})_{n-1}\,(r+n-1)_{n-1}\,((4r-6)\,\omega+r+2n-2)\,\hat{Z}_{2r-2}^{(n)}(\kappa) \\ &\quad +16^{n-1}(4r-6)(r-\tfrac{1}{2})_{n-1}^2\,\hat{Z}_{2r-4}^{(n)}(\kappa) = 0 \end{aligned} \tag{4.38}$$

which is now a recurrence in the length $2r$ for all fixed numbers of walks n. It is this recurrence, which we call the "network-recurrence" that is the generalisation of (3.13) to all n.

4.3. Product Form at κ = 2

When $\kappa = 2$ and so $\omega = 1/4$ the network-recurrence can be solved to find a "product form" for the partition function as in (4.35) which is valid for $\kappa = 1$. To see this we write our network-recurrence for general κ as

$$\begin{aligned} &\omega a_{r-1}^{n} a_r^{n-1} \hat{Z}_{2r}^{(n)}(\kappa) - 4^{n-1}(4\omega a_r^{n-1} b_{r-1}^{n} + a_{r-1}^{n} b_r^{n-1})\,\hat{Z}_{2r-2}^{(n)}(\kappa) \\ &\quad + 16^{n-1} 4 b_{r-1}^{n} b_r^{n-1}\,\hat{Z}_{2r-4}^{(n)}(\kappa) = 0 \end{aligned} \tag{4.39}$$

where

$$a_r^n = (r+n)_n \qquad \text{and} \qquad b_r^n = (r-\tfrac{1}{2})_n \tag{4.40}$$

Substituting $\omega = 1/4$ and defining

$$A_r^n = 2^{2rn} \hat{Z}_{2r}^{(n)}(2) \tag{4.41}$$

we have

$$A_r^n - \left(\frac{b_{r-1}^n}{a_{r-1}^n} + \frac{b_r^{n-1}}{a_r^{n-1}} \right) A_{r-1}^n + \frac{b_{r-1}^n}{a_{r-1}^n} \frac{b_r^{n-1}}{a_r^{n-1}} A_{r-2}^n = 0 \tag{4.42}$$

By inspection of this equation one might guess that $A_r^n / A_{r-1}^n \propto b/a$ without being able to guess the super and sub-scripts required. However, by examining the $n=1$ and $n=2$ solutions[3] one can conjecture

$$A_r^n = \frac{b_r^n}{a_{r-1}^n} A_{r-1}^n \tag{4.43}$$

and it is then a simple matter to show that this satisfies the recurrence. This result implies

$$\hat{Z}_{2r}^{(n)}(2) = 4^n \frac{(r - \frac{1}{2})_n}{(r+n-1)_n} \hat{Z}_{2r-2}^{(n)}(2) \tag{4.44}$$

and knowing the initial condition $\hat{Z}_0^{(n)}(2) = 2$, solving the recurrence gives

$$\hat{Z}_{2r}^{(n)}(2) = 2^{2rn+1} \prod_{j=1}^{r} \frac{(j-\frac{1}{2})_n}{(j+n-1)_n} = 2^{2rn+1} \prod_{i=1}^{n} \frac{(i-\frac{1}{2})_r}{(n+i-1)_r} \tag{4.45}$$

The product form (4.45) derived above and the $\kappa = 1$ product form (4.35) can be asymptotically analysed to give

$$\hat{Z}_{2r}^{(n)}(1) \sim B_{n,o} \mu_{n,o}^{2r} r^{g_{11,o}^{(n)}} \qquad \text{as} \quad r \to \infty \tag{4.46}$$

with

$$\mu_{n,o} = 2^n \qquad \text{and} \qquad g_{11,o}^{(n)} = -\frac{n(2n+1)}{2} \tag{4.47}$$

and

$$\hat{Z}_{2r}^{(n)}(2) \sim B_{n,s} \mu_{n,s}^{2r} r^{g_{11,s}^{(n)}} \qquad \text{as} \quad r \to \infty \tag{4.48}$$

with

$$\mu_{n,s} = 2^n \qquad \text{and} \qquad g_{11,s}^{(n)} = -\frac{n(2n-1)}{2} \tag{4.49}$$

The continuity of the free energy discussed above implies that $\mu_n = 2^n$ for $0 < \kappa \leqslant 2$ at least.

To derive the $\kappa = 2$ result, expressing (4.45) in terms of Γ functions

$$\hat{Z}_{2r}^{(n)}(2) = 2^{2rn+1} \prod_{i=1}^{n} \frac{\Gamma(r+i-\frac{1}{2})\,\Gamma(i+n-1)}{\Gamma(r+n+i-1)\,\Gamma(i-\frac{1}{2})} \tag{4.50}$$

and using Stirling's formula to show that $(\Gamma(r+i-\frac{1}{2})/\Gamma(r+n+i-1)) \sim r^{-(n-1/2)}$ gives (4.48) where

$$B_{n,s} = 2 \prod_{i=1}^{n} \frac{\Gamma(i+n-1)}{\Gamma(i-\frac{1}{2})} = 4(\pi^{-1/2} 2^{n-2})^n \prod_{i=1}^{n} (2i-2)! \tag{4.51}$$

For comparison we give the result for $\kappa = 1$ which is

$$B_{n,o} = (\pi^{-1/2} 2^{n-1})^n \prod_{i=1}^{n} (2i-1)! = \frac{1}{2} \frac{(2n-1)!}{(n-1)!} B_{n,s} \tag{4.52}$$

$$= 4^{n-1} \left(\frac{1}{2}\right)_n B_{n,s} \tag{4.53}$$

4.4. Use of Recurrence Relations and the Phase Diagram

Both the bootstrap-recurrence (4.30) and network-recurrence (4.38) relations yield useful information when the dominant balance method is applied to them. Firstly one assumes a form (4.24) is true at all κ as it is true for $\kappa = 1$ and $\kappa = 2$ from the analyses above, and substitutes this into the recurrence in question. One then imposes a consistency principle to extract information. Application of dominant balance to the network-recurrence (4.38) gives

$$\mu_n = 2^n \qquad \text{or} \qquad 2^{n-1}/\sqrt{\omega} \tag{4.54}$$

to exponential order. Then by looking at corrections at the order $1/r$ it then implies that if $\omega \neq 1/4$ and $\mu_n = 2^n$ then

$$g_{11}^{(n)} = -\frac{n(2n+1)}{2} \tag{4.55}$$

which implies $g_{11}^{(n)} = g_{11,o}^{(n)}$. Rather, if $\omega \neq 1/4$ and $\mu_n = 2^{n-1}/\sqrt{\omega}$ then

$$g_{11}^{(n)} = -\frac{(n-1)(2n-1)}{2} \tag{4.56}$$

Given that the free energy (and so μ_n) must be a continuous function of κ with at least one singularity then the only possible solution consistent with the rigorous bounds discussed above is

$$\mu(\kappa) = \begin{cases} 2^n & \text{if} \quad 0 < \kappa \leqslant 2 \\ 2^{n-1}/\sqrt{\omega} & \text{if} \quad \kappa > 2 \end{cases} \tag{4.57}$$

and hence combining the dominant balance arguments, rigorous results and our $\kappa = 1$ and $\kappa = 2$ analytic results gives

$$g_{11}^{(n)} = \begin{cases} -\dfrac{n(2n+1)}{2} & \text{if} \quad 0 < \kappa < 2 \\ -\dfrac{n(2n-1)}{2} & \text{if} \quad \kappa = 2 \\ -\dfrac{(n-1)(2n-1)}{2} & \text{if} \quad \kappa > 2 \end{cases} \tag{4.58}$$

We have hence calculated the free energy of our surface-watermelons as

$$f^{(n)}(\kappa) = \begin{cases} n \log(2) & \text{if} \quad 0 < \kappa \leqslant 2 \\ \log(\omega(\kappa)^{-1/2}) + (n-1) \log(2) & \text{if} \quad \kappa > 2 \end{cases} \tag{4.59}$$

noting that $\sqrt{\omega(\kappa)} \to 1/2$ as $\kappa \to 2$ so that $f^{(n)}(\kappa)$ is indeed continuous. In other words, it simply differs from single walk results (see Fig. 2) by the constant $(n-1)\log(2)$.

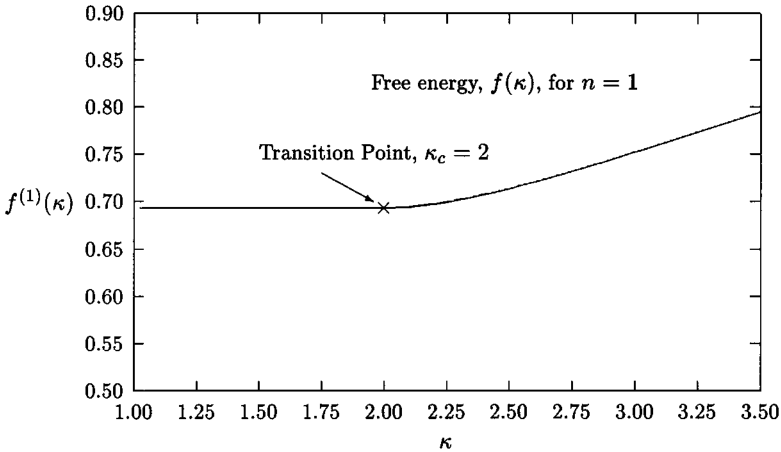

Fig. 2. A plot of the free energy, $f^{(1)}(\kappa)$, of a single walk. The free energy of the n-walk problem differs from this by a constant $(n-1)\log(2)$.

It is of some interest to note that our exponent results for $\kappa \leqslant 2$ can be summarised by a conjectural generalisation of Eq. (3) of ref. 19 (and results from ref. 11) as

$$g_{o,s} = -\tfrac{1}{2}\left(n - V + V_s + \sum_{L \geqslant 1} n_L^b \eta^b + \sum_{L \geqslant 1} n_L^{o,s} \eta^{o,s}\right) \tag{4.60}$$

where n_L^b is the number of terminating L-leg vertices in the bulk and $\eta^b = L(L-1)/2$, n_L^o is the number of terminating L-leg vertices attached to the wall with "ordinary" boundary conditions ($\kappa < 2$) and $\eta^o = L^2$, n_L^s is the number of terminating L-leg vertices attached to the wall with "special" boundary conditions ($\kappa = 2$) and $\eta^s = L(L-1)$, V is the total number of vertices while V_s is the total number of surface vertices. Hence for our surface-watermelons $V = V_s = 2$ while $n_L^b = 0$ for all L. When $\kappa < 2$ $n_n^o = 2$ while $n_L^o = 0$ for $L \neq n$ and $n_L^s = 0$ for all L. When $\kappa = 2$ $n_n^s = 2$ while $n_L^s = 0$ for $L \neq n$ and $n_L^o = 0$ for all L. It is interesting to compare this to the conformal invariance expressions for undirected walks such as ref. 20. Also, on physical grounds one would assume that for $\kappa \gg 2$ the walk closest to the wall is essentially completely stuck to the wall and so $g_{11}^{(n)}(\kappa > 2) = g_{11}^{(n-1)}(\kappa < 2)$, which is indeed the case.

Our conjectured formula enables the following prediction for a watermelon network attached to the surface at one end only with the other free to move in the bulk.

$$g_1^{(n)} = \begin{cases} -\dfrac{3n^2 + n - 2}{4} & \text{if} \quad 0 < \kappa < 2 \\[2ex] -\dfrac{(n-1)(3n+2)}{4} & \text{if} \quad \kappa = 2 \\[2ex] -\dfrac{(n-1)(2n-1)}{2} & \text{if} \quad \kappa > 2 \end{cases} \tag{4.61}$$

The first of these formulae was originally derived by Forrester[4] and the others are in agreement with the cases $n = 1$ and 2 of ref. 3. The last of the formulae is based on our previous argument that for $\kappa \gg 2$ the chain closest to the wall is stuck to the wall so it makes no difference whether the end of the network is attached or not, thus $g_1^{(n)}(\kappa > 2) = g_{11}^{(n)}(\kappa > 2)$.

Lastly, for the sake of comparison let us also apply the dominant balance method to the bootstrap-recurrence (4.30). One finds that if $\mu_n = 2^n$ that

$$g_{11}^{(n)} = g_{11,o}^{(n)} \qquad \text{or} \qquad g_{11}^{(n)} = g_{11}^{(n-1)} + g_{11,o}^{(n-1)} - g_{11,o}^{(n-2)} - 1 \tag{4.62}$$

which implies

$$g_{11}^{(n)} = g_{11,o}^{(n)} \qquad \text{or} \quad g_{11}^{(n)} = -\frac{n(2n-1)}{2} \tag{4.63}$$

This second value we can recognise as $g_{11,s}^{(n)}$ (the $\kappa = 2$ value). It also implies that if $\mu_n \neq 2^n$ then

$$\mu_n = 2^{n-1} \qquad \text{and} \qquad g_{11}^{(n)} = g_{11,o}^{(n)} = -\frac{(n-1)(2n-1)}{2} \tag{4.64}$$

but since we know μ_1 from (3.17), and hence know that $\mu_n \neq 2^n$ for $\kappa > 2$ only, we can deduce that μ_n is given by the same formula (4.57) as derived from the network-recurrence and rigorous arguments. However since $\mu_n = 2^n$ implies one of two exponents one would need to invoke universality to deduce that the exponent is constant if the free energy is analytic and only changes at points of non-analyticity to give the complete solution. So while the bootstrap-recurrence doesn't require the (rigorous) continuity arguments to give μ as a function of κ as did the network-recurrence analysis it does require universality to imply the value of the exponents. If we did not have the $\kappa = 2$ product form results the two dominant balance analyses could be used together to give us the same conclusions for the free energy and exponents.

4.5. Scaling Functions

Certain bijections[13] have allowed the calculation of an expansion in the variable ω for the n-walk partition function $\hat{Z}_{2r}^{(n)}$ similar to (3.9). This expansion

$$\hat{Z}_{2r}^{(n)}(\kappa) = \hat{Z}_{2r}^{(n)}(1)\, \kappa^{-2(n-1)} \sum_{s=0}^{\infty} f_r^{(n)}(s)\, \omega^s \tag{4.65}$$

where

$$f_r^{(n)}(s) = \binom{n+s-1}{s} \frac{(2r+2n-1)_{2s}}{(r+n)_s\,(r+2n)_s} \tag{4.66}$$

is a "formal" expansion and it can be seen by comparison to the $n = 1$ result (3.9) to be valid only when $\kappa \leqslant 2$. It can however be extended to $\kappa > 2$ using the $n = 1$ result as a guide. Here however we are interested in the calculation of a scaling function around $\kappa = 2$ and so all we need do is to calculate the scaling function for $\kappa \leqslant 2$ and then find its analytic continuation to $\kappa > 2$. This should give us an entire function,[15, 3] and this method should be equivalent to first "guessing" the complete solution and later calculating the scaling function (see Section 3.6 of ref. 3) from this complete solution.

We begin by changing the "dummy" summed-over variable to $t = s + r$ so that defining

$$R_r^{(n)}(\kappa) = \kappa^{2(n-1)} \frac{\hat{Z}_{2r}^{(n)}(\kappa)}{\hat{Z}_{2r}^{(n)}(1)} \tag{4.67}$$

we have

$$R_r^{(n)}(\kappa) = \omega^{-r} \sum_{t=r}^{\infty} f_r^{(n)}(t-r)\, \omega^t \tag{4.68}$$

Next we rewrite

$$f_r^{(n)}(t-r) = \frac{1}{(n-1)!}\,(t-r+1)_{n-1}\, \frac{g_t}{g_r} \tag{4.69}$$

where

$$g_s = \frac{\Gamma(2s+2n-1)}{\Gamma(s+n)\,\Gamma(s+2n)} \tag{4.70}$$

Now we have calculated that

$$g_s \sim \frac{4^{s+n-1}}{\sqrt{\pi}}\, s^{-n-1/2} \qquad \text{as} \quad s \to \infty \tag{4.71}$$

This gives us part of what we require to simplify $f_r^{(n)}(t-r)$ in the large r limit. Now we need to consider the other factor $(t-r+1)_{n-1}$. Since we are only interested in the dominant asymptotics of $R_r^{(n)}$ we consider the terms in the expansion of $(t-r+1)_{n-1}$ that give the largest power of $t^a r^b$, that is, $a+b$ is largest. This largest power is $n-1$. Hence we write

$$(t-r+1)_{n-1} = \prod_{j=1}^{n-1} (t-r+j) = (t-r)^{n-1} + O(t^a r^b \ni a+b = n-2) \tag{4.72}$$

Substituting (4.71) and (4.72) into (4.69) and the result into (4.68) gives

$$R_r^{(n)}(\kappa) \sim \sum_{t=r}^{\infty} (4\omega)^{t-r} \left(\frac{r}{t}\right)^{n+1/2} (t-r)^{n-1} \tag{4.73}$$

and, with $u = t/r$, as $r \to \infty$ for $\kappa \leqslant 2$, replacing the sum by an integral gives

$$R_r^{(n)}(\kappa) \sim \frac{r^n}{(n-1)!} \int_1^\infty e^{-(r \log 1/4\omega)(u-1)} (u-1)^{n-1}\, u^{-(n+1/2)}\, du \tag{4.74}$$

Note that this expression is uniform in κ for $\kappa \leqslant 2$ and the corrections are of order $1/r$. To obtain the scaling function valid as $\kappa \to 2$ we make the approximation

$$\log \frac{1}{4\omega} \approx \left(\frac{\kappa - 2}{2}\right)^2 \tag{4.75}$$

Hence with a change to the scaling variable (3.23) $v = ((\kappa - 2)/2)\, r^{1/2}$ used in the single walk case we can write, as $\kappa \to 2^-$ and $r \to \infty$

$$R_r^{(n)}(\kappa) \approx \frac{r^n}{(n-1)!} \int_1^\infty e^{-v^2(u-1)} (u-1)^{n-1}\, u^{-(n+1/2)}\, du = r^n U\left(n, \frac{1}{2}, v^2\right) \tag{4.76}$$

where $U(a, b, y)$ is a confluent hypergeometric function and is one of the solutions to Kummer's differential equation (see p. 504 of ref. 21).

While elegant this result is only valid for $v \leqslant 0$ but, expanding the integrand,

$$R_r^{(n)}(\kappa) \approx \frac{r^n}{(n-1)!} \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{n-1}{k} \chi\left(v^2, n-k+\frac{1}{2}\right) \quad \text{as} \quad r \to \infty \tag{4.77}$$

where

$$\chi(y, b) = \int_1^\infty e^{-y(u-1)} \frac{du}{u^b} \tag{4.78}$$

By applying an easily derivable recurrence for the integrals $\chi(y, b)$ and noting that this integral with $b = \frac{1}{2}$ can be written in terms of the complementary error function which is an entire function (and so provides the required analytic continuation to $v > 0$),

$$\chi\left(v^2, \frac{1}{2}\right) = -\frac{1}{v}\, e^{v^2} \operatorname{erfc}(-v) \tag{4.79}$$

one finds

$$\hat{Z}_{2r}^{(n)}(\kappa) \sim \frac{2^{2nr}}{r^{n(2n-1)/2}} \hat{\varphi}^{(n)}\left(\frac{(\kappa-2)}{2} r^{1/2}\right) \qquad \text{as} \quad \kappa \to 2 \quad \text{and} \quad r \to \infty \tag{4.80}$$

with

$$\hat{\varphi}^{(n)}(v) = C^{(n)} \left(p_a(v) + \frac{\sqrt{\pi}}{v} e^{v^2} \operatorname{erfc}(-v)\, p_b(v)\right) \tag{4.81}$$

where the polynomials $p_a(v)$ and $p_b(v)$ are

$$p_a(v) = \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \sum_{\ell=1}^{j+1} (-1)^{\ell-1} \frac{v^{2\ell-2}}{(j-\ell+\frac{3}{2})_\ell} \tag{4.82}$$

and

$$p_b(v) = \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{v^{2j+2}}{(\frac{1}{2})_{j+1}} \tag{4.83}$$

The multiplying constant

$$C^{(n)} = \frac{B_{n,o}}{4^{n-1}\Gamma(n)} = \left(\frac{1}{2}\right)_n \frac{B_{n,s}}{\Gamma(n)} \tag{4.84}$$

is given in terms of $B_{n,o}$, and hence in terms of $B_{n,s}$, from the asymptotics of the $\kappa = 1$ partition function (4.53). This result should now be valid for all v as we have analytically continued (or rather "uniformised") the scaling function to give us an entire function. The definition of the scaling variable v implies that the crossover exponent $\phi = 1/2$ for all n.

Using the property $\operatorname{erfc}(-v) = 2 - \operatorname{erfc}(v)$ in (4.81) leads to

$$\hat{\varphi}^{(n)}(-v) = \hat{\varphi}^{(n)}(v) - \frac{2C^{(n)}\sqrt{\pi}\, e^{v^2}}{v} p_b(v) \tag{4.85}$$

and hence in terms of Kummer's function, for all v

$$\hat{\varphi}^{(n)}(v) = \left(\frac{1}{2}\right)_n B_{n,s} \left(U\left(n, \frac{1}{2}, v^2\right) + \theta(v) \frac{2\sqrt{\pi}\, e^{v^2}}{\Gamma(n)\, v} p_b(v)\right) \tag{4.86}$$

The behavior of the scaling function (4.81) as $v \to \pm\infty$ matches the fixed κ results, as expected from the use of the symbol $\approx$,[15] with

$$\hat{\varphi}^{(n)}(v) \sim \begin{cases} C^{(n)} \Gamma(n)\, v^{-2n} = \left(\frac{1}{2}\right)_n B_{n,s} v^{-2n} & \text{for} \quad v \to -\infty \\ C^{(n)} \sum_{j=0}^{n-1} \frac{(-1)^j}{(j+\frac{1}{2})} \binom{n-1}{j} = B_{n,s} & \text{for} \quad v = 0 \\ \frac{2\pi C^{(n)}}{\Gamma(n+\frac{1}{2})} v^{2n-1} e^{v^2} = \frac{2\sqrt{\pi}}{\Gamma(n)} B_{n,s} v^{2n-1} e^{v^2} & \text{for} \quad v \to \infty \end{cases} \tag{4.87}$$

The case $v \to \infty$ comes from the second term of (4.86) noting that $p_b(v)$ is dominated by the term v^{2n} and the other cases follow from the asymptotic forms of Kummer's function as $v \to -\infty$ ([21, Eq. 13.5.2]) and $v \to 0$ ([21, Eq. 13.5.10]). The first of these arises by rewriting the integral as

$$U\left(n, \frac{1}{2}, v^2\right) = \frac{v^{2n}}{\Gamma(n)} \int_0^\infty e^{-x} x^{n-1} \left(1 + \frac{x}{v^2}\right)^{-(n+1/2)} dx \tag{4.88}$$

and using the binomial expansion.

Note firstly that our $v = 0$ result is equivalent to the result (4.51) derived from the product form. Also, note that there is an error in the results of ref. 3 for the case $n = 2$: Eq. (4.51) should have an extra factor of κ which means that $\hat{\phi}^{\mathscr{V}}$ should have an extra factor of 2. This scaling function (or rather set of scaling functions) derived above provides a full description of the dominant asymptotics around the adsorption transition.

5. SUMMARY OF RESULTS

In this paper we have analysed the scaling behavior of n directed walks interacting with a "wall" via a contact potential, with Boltzmann factor κ, in the specific case of a watermelon conformation tied to the surface at both ends, as described in Section 2. We have provided a summary of rigorous results for the thermodynamic-limit free energy that can easily be derived for this model (Section 4.1). We have found a recurrence relation (Eq. (4.38)) for each n that the partition function satisfies. We have used this recurrence to derive a product form for the partition function at $\kappa = 2$ which is the value of κ at which there is an adsorption transition of the network to the surface. The network exponent (and free energy) at this value has also been calculated (Eq. (4.49)). Combining the rigorous results and the $\kappa = 2$ results with a dominant balance analysis of the said recurrence leads to the calculation of the free energy and network exponents at all fixed κ (Eqs. (4.57) and (4.58) respectively). Importantly we have used an expansion of the partition function in a different variable to derive a full scaling function (Eqs. (4.80), (4.81), (4.82) and (4.83)) around the adsorption transition for every value of n.

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